Tentamen Partiële Differentiaalvergelijkingen
1 Juli 2010, 14.00–17.00 uur

Geef per vraagstuk duidelijk aan wat het antwoord is; bereken in voorkomende gevallen de optredende Fouriercoefficienten expliciet. Geef steeds duidelijk aan welke oplossingsmethode gebruikt wordt.

1. Bepaal de oplossing van het beginwaarde-probleem

$$u_t + (\log u)u_x = 0, \quad u(x,0) = e^x, \quad x \in \mathbb{R}, \quad t > 0.$$

2. Geef de oplossing van

$$u_t = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$$

als de volgende voorwaarden worden gesteld

$$u(0,t) = 0, \quad u_x(L,t) = 0, \quad u(x,0) = f(x).$$

3. Geef de oplossing van

$$u_t = c^2 u_{xx} + q(x,t), \quad t > 0, \quad 0 < x < L,$$

als de volgende voorwaarden gesteld worden:

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0, \quad u(x,0) = f(x).$$

4. Geef een voorbeeld van harmonische functies u en v waarvoor het product uv niet harmonisch is. Toon aan dat als u, v, en $u^2 + v^2$ harmonisch zijn op een open samenhangend gebied, dan u en v constant zijn.

5. Bepaal met behulp van de Fourier transformatie de oplossing van

$$u_t = \frac{1}{100} u_{xx}, \quad u(x,0) = 100, \quad |x| < 1, \quad u(x,0) = 0, \quad \text{elders.}$$

6. Bepaal de oplossing $u(x,t)$, $0 \le x \le \pi$, $t > 0$, van

$$u_{tt} + u_t = u_{xx}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = 0.$$

Hoe ziet de oplossing eruit als $f(x) = \sin x$?

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1 $x_0(s) = s$
$t_0(s) = 0$ parametrisering
$\tilde{u}_0(s) = e^s$

$\frac{\partial x}{\partial \tau}(\tau,s) = \log(u)$ $x(0,s) = s$
$\frac{\partial t}{\partial \tau}(\tau,s) = 1$ $t(0,s) = 0 \rightarrow t(\tau,s) = \tau$
$\frac{\partial \tilde{u}}{\partial \tau}(\tau,s) = 0$ $u(0,s) = e^s \rightarrow u(\tau,s) = e^s$

$\left.\begin{array}{l}\frac{\partial x}{\partial \tau} = \log(u) = \log(e^s) = s \\ x(0,s) = s\end{array}\right\} \rightarrow x(\tau,s) = s\tau + s = s(\tau+1)$

$\frac{x}{t+1} = \frac{s(\tau+1)}{\tau+1} = s$

$\tilde{u}(\tau,s) = e^s = u(x,t) = e^{\frac{x}{t+1}}$

controle:
$u(x,0) = e^x$
$\left.\begin{array}{l}\frac{\partial u}{\partial t} = -\frac{x}{(t+1)^2} e^{\frac{x}{t+1}} = u_t \\ \frac{\partial u}{\partial x} = u_x = \frac{1}{t+1} e^{\frac{x}{t+1}} \\ \log(u) = \log(e^{\frac{x}{t+1}}) = \frac{x}{t+1}\end{array}\right\} \rightarrow u_t + \log(u)u_x = 0$

2 stel $u(x,t) = X(x)T(t)$ separatie van variabelen

$T'(t)X(x) = c^2 X''(x)T(t)$

$\frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = k \ \forall \ t>0, \ 0<x<l$, dus moet dit constant zijn

zie (*)
Stel $\underline{k>0}$ $k = \mu^2$ $(\mu>0)$
$\rightarrow X''(x) - \mu^2 X(x) = 0$ levert de algemene oplossing
$X(x) = c_1 \cosh(\mu x) + c_2 \sinh \mu(x)$

(*) $X(0)\,T(t) = 0 \;\forall t$ } om triviale oplossing ($T=0$) te
$X(L)\,T(t) = 0 \;\forall t$ voorkomen $X(0) = 0$ en $X(L) = 0$

$k>0$:
dus $X(0) = c_1 = 0$
$X(L) = c_2 \sinh(\mu L)$
$\sinh(\mu L) \neq 0 \;\forall \mu$ ($\mu \neq 0$ want $k>0$)
dus dit levert alleen de triviale oplossing
$X = 0$

$k=0$: $X''(x) = 0$
$X(x) = c_1 x + c_2$
$X(0) = c_2 = 0$
$X(L) = c_1 L = 0 \to c_1 = 0$
Dus weer alleen de triviale oplossing

$k<0$: $k = -\mu^2$ $(\mu > 0)$
$X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$
$X(0) = c_1 = 0$ $\to c_2 = 1$ (om triv. opl. te voorkomen)
$X(L) = \sin(\mu L) = 0$
$\mu L = n\pi$ $n = \pm 1, \pm 2, \ldots$
$\mu_n = \frac{n\pi}{L}$
$X(x) = \sin\left(\frac{n\pi}{L} x\right)$

$\frac{T'(t)}{T(t)} = -\mu^2 \to T'(t) + \mu^2 T(t) = 0$

$T(t) = c\, e^{-\mu^2 t}$

$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right)$

$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right)$ Fourierontwikkeling van f

$c_n = \frac{2}{L} \int f(x) \sin\left(\frac{n\pi}{L} x\right) dx$

4 ω harmonisch als

$$\nabla^2 \omega = \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} = 0$$

Stel $u = x$

$v = x$

Dan $\nabla^2 u = 0$

$\nabla^2 v = 0$

$\nabla^2 (uv) = \nabla^2 (x^2) = 2 \neq 0$ niet harmonisch

Stel $u, v, u^2 + v^2$ harmonisch

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\nabla^2 (u^2+v^2) = \frac{\partial^2 (u^2+v^2)}{\partial x^2} + \frac{\partial^2 (u^2+v^2)}{\partial y^2} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + \frac{\partial^2 v^2}{\partial x^2} + \frac{\partial^2 v^2}{\partial y^2} = 0$$

$$= \frac{\partial}{\partial x}\left(2u \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(2u \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial x}\left(2v \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(2v \frac{\partial v}{\partial y}\right)$$

$$= 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u \frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial y}\right)^2 + 2u \frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial v}{\partial x}\right)^2 + 2v \frac{\partial^2 v}{\partial x^2} + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2v \frac{\partial^2 v}{\partial y^2}$$

$$= 2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) + 2u\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{=0} + 2\left(\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right) + 2v\underbrace{\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}_{=0}$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$a^2 \geq 0 \quad a \in \mathbb{R}$ dus

$\left(\frac{\partial u}{\partial x}\right)^2 \geq 0, \left(\frac{\partial u}{\partial y}\right)^2 \geq 0, \left(\frac{\partial v}{\partial x}\right)^2 \geq 0, \left(\frac{\partial v}{\partial y}\right)^2 \geq 0$

$\nabla^2 (u^2+v^2) = 0$ dus

$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0$, en

dus u, v constant.

5 $\frac{\partial \hat{u}(\omega,t)}{\partial t} = -\frac{\omega^2}{100}\hat{u}(\omega,t) = \frac{1}{100}\frac{\partial^2}{\partial x^2}\hat{u}(t,\omega) = \frac{(i\omega)^2}{100}\hat{u}(t,\omega)$

~~$\frac{\partial \hat{u}(t,\omega)}{\partial t}$~~ ~~$\hat{u}(t,\omega) = c e^{-\frac{\omega^2}{100}t}$~~

$\hat{u}(\omega,t) = A(\omega) e^{-\frac{\omega^2}{100}t}$

$\hat{u}(\omega,0) = 100 \qquad |\omega| < 1$, anders $u(x,0) = 0$

~~$\hat{u}(\omega,0) = A(\omega)$~~ ~~$A(\omega) = \begin{cases} 100 & |\omega|<1 \\ 0 & \text{anders} \end{cases}$~~

$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\omega) e^{-\frac{\omega^2}{100}t} e^{i\omega x} d\omega$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\omega) e^{-\frac{\omega^2}{100}t} e^{i\omega x} d\omega$

controle:

$u_t = \frac{\partial u}{\partial t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\omega) \cdot \frac{-\omega^2}{100} e^{-\frac{\omega^2}{100}t} e^{i\omega x} d\omega$

$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -\omega^2 \cdot A(\omega) e^{-\frac{\omega^2}{100}t} e^{i\omega x} d\omega$

dus $u_t = \frac{1}{100} u_{xx}$

$u(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\omega) e^{i\omega x} d\omega$

$= \frac{1}{\sqrt{2\pi}} \frac{100}{ix} e^{i\omega x} \Big|_{-1}^{1}$

$= \frac{1}{\sqrt{2\pi}} \frac{100}{ix} [e^{ix} - e^{-ix}]$

$= \frac{100}{\sqrt{2\pi}} \cdot 2\sin(x)$

3 ~~oplossing totdeel~~

zie 2:
alleen nu
$X(L) = \sin(\mu L) = 0$
$\mu_n = \frac{n\pi}{L}$

Dus $u_1(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)$

$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)$ voldoet aan homogene deel

$\frac{u_t}{q} = \frac{c^2 u_{xx}}{q} + 1 = k$

$u_t = kq$ $\qquad \frac{c^2 u_{xx}}{q} = k - 1$

$u = \int kq \, dt$

$x_0(s) = s$ $\qquad \frac{\partial^2 x}{\partial \tau^2} = c^2$ $\qquad x(\tau,s) = \frac{1}{2}c^2\tau^2 + M\tau + s$

$t_0(s) = 0$ $\qquad \frac{\partial t}{\partial \tau} = 1$ $\qquad t = \tau$

$u_0(s) = f(s)$ $\qquad \frac{\partial u}{\partial \tau} = q(x,t)$ $\qquad u = q(x,t) + f(s)$

$u(\tau, s) =$

$u(\tau, 0) = 0$

$u(\tau, L) = 0$

6 separatie van variabelen

Stel $u(x,t) = X(x)T(x)$

$T''(t)X(x) + T'(t)X(x) = X''(x)T(t)$

$$\frac{T''(t)+T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\mu^2 \quad \text{(zie opg 2)} \quad \mu > 0$$

$X(0)=0$ en $X(\pi)=0$ (anders $T(t)=0\ \forall t \geq 0$: triviale oplossing)

$X(x) = c_1\cos(\mu x) + c_2\sin(\mu x)$

$X(0) = c_1 = 0 \Rightarrow c_2 = 1$ (voorkomt triviale oplossing)

$X(\pi) = \sin(\mu\pi)$

$\mu = n \quad n = \pm 1, \pm 2, \dots \quad X_n(x) = \sin(nx)$

$T''(t) + T'(t) + \mu^2 T(t) = 0$

probeer $T(t) = e^{\lambda t}$ dit levert de karakteristieke vergelijking $\lambda^2 + \lambda + \mu^2 = 0$

$$\lambda_\pm = \frac{-1 \pm \sqrt{1^2 - 4\mu^2}}{2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \mu^2}$$

als $\mu = \frac{1}{2}$, dan $\lambda_1 = \lambda_2 = -\frac{1}{2}$

dus dan wordt de oplossing:

$T_0(t) = a_1 e^{-\frac{1}{2}t} + a_2 t e^{-\frac{1}{2}t}$ maar kan niet $\mu = n \in \mathbb{Z}$

dus $n=0$ → valt weg want $X_0(x) = 0$

als $\mu < \frac{1}{2}$: $\mu^2 < \frac{1}{4}$ dus $\frac{1}{4} - \mu^2 > 0$

dan

$T_0(t) = b_1\cos(\lambda_0 t) + b_2\sin(\lambda_0 t)$

dus $n > 0$

als $\mu > \frac{1}{2}$: $\frac{1}{4} - \mu^2 < 0$ dan

$T_n(t) = d_1\cosh(\lambda t) + d_2\sinh(\lambda t)$

~~$u(x,t) = e^{-\frac{1}{2}t} + a_2 t e$~~

$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi)\left[d_1\cosh(\lambda_n t) + d_2\sinh(\lambda_n t)\right]$

$$\lambda_{n1} = -\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - n^2}$$
$$\lambda_{n2} = -\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - n^2}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\,[d_1 \cosh(\lambda_{n1} t) + d_2 \sinh(\lambda_{n1} t)]$$
$$+ \sum_{n=1}^{\infty} \sin(nx)\,[h_1 \cosh(\lambda_{n2} t) + h_2 \sinh(\lambda_{n2} t)]$$

$$u(x,0) = \sum_{n=1}^{\infty} d_1 \sin(nx) + \sum_{n=1}^{\infty} h_1 \sin(nx) = f(x)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin(nx)\,\lambda_{n1} d_2 + \sum_{n=1}^{\infty} \sin(nx)\,\lambda_{n2} h_2 = 0$$

$\rightarrow d_2 = 0$ en $h_2 = 0$ (want dit moet gelden $\forall n$)